# On calculating the mean values of quantum observables in the optical tomography representation

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**Abstract**

Given a density operator $\hat{\rho}$ the optical tomography map defines a one-parameter set of probability distributions $w_{\hat{\rho}}(X,\phi)$, $\phi\in[0,2\pi)$, on the real line allowing to reconstruct $\hat{\rho}$. We introduce a dual map from the special class $\mathcal{A}$ of quantum observables $\hat{a}$ to a special class of generalized functions $a(X,\phi)$ such that the mean value $<\hat{a}>_{\hat{\rho}}=Tr(\hat{\rho}\hat{a})$ is given by the formula $<\hat{a}>_{\hat{\rho}}=\int\limits_0^{2\pi}\int\limits_{-\infty}^{+\infty}w_{\hat{\rho}}(X,\phi)a(X,\phi)dXd\phi$. The class $\mathcal{A}$ includes all the symmetrized polynomials of canonical variables $\hat{q}$ and $\hat{p}$.

## 1 Introduction

Given an observable (hermitian operator) $\hat{a}$ in a Hilbert space $H$ the spectral theorem reads

$$\hat{a}=\int\limits_{\mathbb{R}}Xd\hat{E}((-\infty,X]),$$

where $\hat{E}$ in an orthogonal projection valued measure defined on all Borel subsets $\Omega\subset\mathbb{R}$ such that $\hat{E}(\Omega)$ is an orthogonal projection and the projections $\hat{E}(\Omega_1)$, $\hat{E}(\Omega_2)$ are orthogonal for all open $\Omega_1,\Omega_2\subset\mathbb{R}$, $\Omega_1\cap\Omega_2=\emptyset$. Using the projection valued (spectral) measure $\hat{E}$ transforms the Hilbert space $H$ to the Hilbert space $H_{\hat{a}}=L^2(\mathbb{R})$ formed by wave functions $\psi_{\hat{a}}(\cdot)$ obtaining from $\psi\in H$ by the formula

$$\psi_{\hat{a}}(X)=\frac{d}{dX}\left(\hat{E}((-\infty,X])\psi\right).$$

The Hilbert space $H_{\hat{a}}$ is said to be a space of representation associated with the observable $\hat{a}$.

Suppose that $\hat{\rho}$ is a density operator (positive unit-trace operator), then in any space of representation $H_{\hat{a}}$ it can be represented as an integral operator

$$(\hat{\rho}\psi_{\hat{a}})(X)=\int\limits_{\mathbb{R}}\rho_{\hat{a}}(X,Y)\psi_{\hat{a}}(Y)dY,$$

$\psi_{\hat{a}}(\cdot) \in H_{\hat{a}}$. In the case, the Hilbert-Schmidt kernel $\rho_{\hat{a}}(\cdot,\cdot)$ is said to be a density matrix of $\hat{\rho}$ in the space of representation $H_{\hat{a}}$. Analogously, one can define the density matrix $b(\cdot,\cdot)$ (which can be a generalized function) associated with a observable $\hat{b}$ in the space of representation $H_{\hat{a}}$.

In [1] the Wigner function $W(q,p)$ associated with the density matrix $\hat{\rho}(\cdot,\cdot)$ in the space of representation associated with the position operator $\hat{q}$ was introduced as

$$W(q,p) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{-ipx} \rho\left(q + \frac{x}{2}, q - \frac{x}{2}\right) dx.$$

The Moyal representation of quantum mechanics [2] defines a map between quantum observables $\hat{a}$ and functions $a(q,p)$ on the phase space under which the mean value $< \hat{a} >_{\hat{\rho}} = Tr(\hat{\rho}\hat{a})$ is given by the formula

$$< \hat{a} >_{\hat{\rho}} = \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} W(q,p) a(q,p) dq dp.$$

Unfortunately, although the normalization rule $\int\int W(q,p)dqdp = 1$ holds, the Wigner function $W(q,p)$ is not positive definite in general. In [3, 4] the optical tomogram $w(X,\phi)$ which can be calculated under experimental measuring a generalized homodyne quadrature was introduced as the Radon transform of the Wigner function,

$$w(X,\phi) = \int\int W(q,p)\delta(X - \cos(\phi)q - \sin(\phi)p) dq dp,$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators. The one-parameter set $\{w(X,\phi),\ \phi \in [0,2\pi)\}$ consists of probability distributions on the real line. The optical tomogram can be calculated from the density operator directly by means of the formula [5]

$$w(X,\phi) = Tr(\hat{\rho}\delta(X - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})).$$

The inverse Radon transform [6] allows to reconstruct the Wigner function from the optical tomogram.

For a density operator $\hat{a}$ one can define a function of complex variable $z$ by the formula

$$a(z,\phi) = -2\pi Tr(\hat{a}(z - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2}), \tag{1}$$

$z \in \mathbb{C},\ Im(z) \neq 0,\ \phi \in [0, 2\pi]$.

In the present paper we shall correct the mistake in [7]. Our goal is to prove the following statements.

**Theorem 1.** *For any density operator $\hat{\rho}$ the following identity holds,*

$$\lim_{\varepsilon \to +0} \int\limits_{0}^{2\pi} \int\limits_{-\infty}^{+\infty} w_{\hat{\rho}}(X + i\varepsilon, \phi) a(X + i\varepsilon, \phi) dX d\phi = Tr(\hat{\rho}\hat{a}).$$

**Definition.** We shall call the relation (1) *a map dual to the optical tomogram map.*

It should be noted that the notion of duality we introduce is different from the known concept of [8].

Denote $\mathcal{D}$ the convex set of density operators whose kernels in the coordinate representation belong to the Schwartz space $S(\mathbb{R}^2)$. Then, optical tomograms corresponding to states from $\mathcal{D}$ belong to the space $\Omega$ consisting of functions $w(X,\phi)$ which are from the Schwartz space in $x$ and infinitely differentiable in $\phi$. Notice that $\mathcal{A} = \mathcal{D}^*$ contains all bounded quantum observables at least.

**Corollary 2.** *The dual map (1) can be extended to any $\hat{a} \in \mathcal{A}$. The extension $a(X,\phi)$ belongs to the adjoint space $\Omega^*$. Moreover, for any density operator $\hat{\rho} \in \mathcal{D}$ the equality*

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} w_{\hat{\rho}}(X,\phi)a(X,\phi)dXd\phi = Tr(\hat{\rho}\hat{a})$$

*holds.*

Let us define a symmetrized product of canonical quantum observables $\hat{q}^m\hat{p}^n$ as

$$\{\hat{q}^m\hat{p}^n\}_s = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k \hat{p}^k\hat{q}^m\hat{p}^{n-k}. \tag{2}$$

Below we use the trigonometric polynomials $Q_n^m(\cos(\phi))$ defined in Appendix.

**Theorem 3.** *The action of the dual map (1) to the observables (2), gives rise to $a_{mn}(X,\phi)$ of the form*

$$a_{mn}(X,\phi) = Q_{n+m}^m(\cos(\phi))X^{n+m}.$$

# 2 The Parseval equality associated with the characteristic functions

Given a density operator $\hat{\rho}$ the function $F(\mu,\nu) = Tr(\hat{\rho}e^{i\mu\hat{q}+i\nu\hat{p}})$ is said to be a characteristic function of the state $\hat{\rho}$. The associated set of probability distributions is said to be *a symplectic quantum tomogram* [5]

$$w(X,\mu,\nu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iXt}F(t\mu,t\nu)dt$$

which is connected with the optical tomogram by the formula

$$w(X,\phi) = w(X,\cos(\phi),\sin(\phi)).$$

In this way,

$$F(t\cos(\phi), t\sin(\phi)) = \int\limits_{-\infty}^{+\infty} e^{itX} w(X,\phi) dX. \tag{3}$$

The standard identity $e^{i\mu\hat{q}+i\nu\hat{p}} = e^{\frac{i\mu\nu}{2}} e^{i\mu\hat{q}} e^{i\nu\hat{p}}$ results in

$$F(\mu,\nu) = \int\limits_{-\infty}^{+\infty} e^{i\mu x} \rho\left(x+\frac{\nu}{2}, x-\frac{\nu}{2}\right) dx. \tag{4}$$

It immediately follows from (4) that the following Parseval-type equality holds,

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |F(\mu,\nu)|^2 d\mu\nu = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \rho(X,Y) dX dY = \frac{1}{2\pi} Tr(\hat{\rho}^2),$$

which is equivalent to

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu,\nu)\overline{F}_{\hat{\sigma}}(\mu,\nu) d\mu\nu = \frac{1}{2\pi} Tr(\hat{\rho}\hat{\sigma}) \tag{5}$$

for the characteristic functions of any two density operators $\hat{\rho}$ and $\hat{\sigma}$.

Taking into account the Parseval-type equality (5) it is possible to extend the map $\hat{\rho} \to F_{\hat{\rho}}$ to all operators of Hilbert-Schmidt class. Moreover, one can construct a tempered distribution $F_{\hat{a}} \in S'(\mathbb{R}^2)$ associated with an observable $\hat{a}$ such that $\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu,\nu)F_{\hat{a}}(\mu,\nu) d\mu d\nu = \frac{1}{2\pi} Tr(\hat{\rho}\hat{a})$ for all density operators $\hat{\rho} \in \mathcal{D}$. The following result is well-known ([2]) and we put it for the sake of completeness.

**Proposition 4.** *The tempered distributions $F_{\hat{a}} \equiv F_{mn}$ associated with the observables $\hat{a}$ of the form (2) are given by the formula*

$$F_{mn}(\mu,\nu) = (-i)^{m+n}\delta^{(m)}(\mu)\delta^{(n)}(\nu).$$

Proof.

Using the Parseval type identity (5) we get

$$\int\limits_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu,\nu)F_{00}(\mu,\nu) d\mu d\nu = \frac{1}{2\pi} Tr(\hat{\rho}) = \frac{1}{2\pi}.$$

Since $F_{\hat{\rho}}(0,0) = 1$ for all density operators $\hat{\rho}$ it results in

$$F_{00}(\mu,\nu) = \delta(\mu)\delta(\nu). \tag{6}$$

Notice that the statement holds if either $m$ or $n$ equals zero. Suppose that it is true for all integer numbers up to fixed $m$ and $n$, let us prove that it holds for $m+1$ and $n+1$. Using the equalities

$$\hat{p}^k\hat{q}^m\hat{p}^{n-k} = \hat{p}^k\hat{q}^{m+1}\hat{p}^{n-k} - i(n-k)\hat{p}^k\hat{q}^m\hat{p}^{n-k-1}$$

and

$$\nu\delta(\nu) = 0,\ \nu\delta^{(n)}(\nu) = -n\delta^{(n-1)}(\nu),\ n \geq 1,$$

we get

$$\frac{\partial}{\partial\mu}\left(Tr\left(\{\hat{q}^m\hat{p}^n\}_s e^{i\mu\hat{q}+i\nu\hat{p}}\right)\right) = Tr\left(\{\hat{q}^m\hat{p}^n\}_s(i\hat{q}+\frac{i\nu}{2})e^{\frac{i\mu\nu}{2}}e^{i\mu\hat{q}}e^{i\nu\hat{p}}\right) =$$

$$iTr\left(\{\hat{q}^m\hat{p}^n\}_s\hat{q}e^{i\mu\hat{q}+i\nu\hat{p}}\right) - \frac{i}{2}n\delta^{(m)}(\mu)\delta^{(n-1)}(\nu) = iTr\left(\{\hat{q}^{m+1}\hat{p}^n\}_s e^{i\mu\hat{q}+i\nu\hat{p}}\right).$$

On the other hand, the equality

$$\frac{\hat{p}}{2}\{\hat{q}^m\hat{p}^n\}_s + \{\hat{q}^m\hat{p}^n\}_s\frac{\hat{p}}{2} = \{\hat{q}^m\hat{p}^{n+1}\}_s$$

results in

$$\frac{\partial}{\partial\nu}\left(Tr\left(\{\hat{q}^m\hat{p}^n\}_s e^{i\mu\hat{q}+i\nu\hat{p}}\right)\right) = Tr(\{\hat{q}^m\hat{p}^n\}_s e^{\frac{i\mu\nu}{2}}e^{i\mu\hat{q}}(i\hat{p}+\frac{i\mu}{2})e^{i\nu\hat{p}}) =$$

$$Tr(\frac{ip}{2}\{\hat{q}^m\hat{p}^n\}_s e^{\frac{i\mu\nu}{2}}e^{i\mu\hat{q}}e^{i\nu\hat{p}}) + Tr(\{\hat{q}^m\hat{p}^n\}_s e^{\frac{i\mu\nu}{2}}(\frac{i\hat{p}}{2}-\frac{i\mu}{2})e^{i\mu\hat{q}}e^{i\nu\hat{p}}) + \frac{im}{2}\delta^{(m-1)}(\mu)\delta^{(n)}(\nu) =$$

$$iTr(\{\hat{q}^m\hat{p}^{n+1}\}_s e^{i\mu\hat{q}+i\nu\hat{p}}).$$

□

# 3 The dual map

To prove Theorem 1 and Corollary 2 we need the following result.

**Proposition 5.** *Given a density operator $\hat{a}$ the relation between the dual map (1) and the characteristic function $F_{\hat{a}}$ is given by*

$$tF_{\hat{a}}(t\cos(\phi), t\sin(\phi)) = \frac{1}{(2\pi)^2}\lim_{\varepsilon\to+0}\int\limits_{-\infty}^{+\infty} e^{itX}a(X-i\varepsilon,\phi)dX,\ t>0.$$

Proof.

Let us consider the representation of $\cos(\phi)\hat{p}+\sin(\phi)\hat{q}$ in the space $H_\phi = L^2(\mathbb{R})$ such that

$$((\cos(\phi)\hat{p}+\sin(\phi)\hat{q})f)(x) = xf(x),\ f\in H_\phi.$$

Then, given $f, g \in H_\phi$

$$\int_{-\infty}^{+\infty} e^{itX}(g,(X - i\varepsilon - \cos(\phi)\hat{p} - \sin(\phi)\hat{q})^{-2} f)dX =$$

$$\int_{-\infty}^{+\infty} \overline{g}(x) f(x) \int_{-\infty}^{+\infty} e^{itX} \frac{1}{(X - x - i\varepsilon)^2} dX dx \equiv I$$

Calculating the residue in $z_0 = x + i\varepsilon$ we obtain

$$I = 2\pi i \left\{ \begin{array}{c} it(g, e^{it(\cos(\phi)\hat{q}+\sin(\phi)\hat{p}+i\varepsilon)} f), \ t > 0 \\ 0, \ t < 0 \end{array} \right.$$

□

Proof of Theorem 1.

Using the expression of $w_{\hat{\rho}}$ through the characteristic function $F_{\hat{\rho}}$ and the definition of $a(z, \phi)$ we obtain

$$\lim_{\epsilon \to +0} \int_0^{2\pi} \int_{-\infty}^{+\infty} w_{\hat{\rho}}(X + i\varepsilon, \phi) a(X + i\varepsilon, \phi) dX d\phi =$$

$$- \lim_{\epsilon \to +0} \int_0^{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-itX} F_{\hat{\rho}}(t\cos(\phi), t\sin(\phi)) Tr(\hat{a}(X + i\varepsilon - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2}) dt dX d\phi =$$

$$- \lim_{\epsilon \to +0} \int_0^{2\pi} \int_{-\infty}^{+\infty} F_{\hat{\rho}}(t\cos(\phi), t\sin(\phi)) \left( \int_{-\infty}^{+\infty} Tr(\hat{a}e^{-itX}(X + i\varepsilon - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2}) dX \right) dt d\phi =$$

$$2\pi \lim_{\epsilon \to 0} \int_0^{2\pi} \int_{-\infty}^{+\infty} F_{\hat{\rho}}(t\cos(\phi), t\sin(\phi) \left( \overline{-\frac{1}{2\pi} \int_{-\infty}^{+\infty} Tr(\hat{a}e^{itX}(X - i\varepsilon - \cos(\phi)\hat{q} - \sin(\phi)\hat{p})^{-2} dX} \right) dt d\phi \equiv I$$

Substituting the relation of Proposition 5 we get

$$I = 2\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_{\hat{\rho}}(\mu, \nu) \overline{F}_{\hat{a}}(\mu, \nu) d\mu d\nu = Tr(\hat{\rho}\hat{a})$$

□

Proof of Corollary 2.

If a density operator $\hat{\rho} \in \mathcal{D}$, i.e. the density matrix in the coordinate representation $\rho(\cdot, \cdot) \in S(\mathbb{R}^2)$, then its characteristic function

$$F_{\hat{\rho}}(\mu, \nu) = \int_{-\infty}^{+\infty} e^{i\mu x} \rho\left(x + \frac{\nu}{2}, x - \frac{\nu}{2}\right) dx$$

is also from the Schwartz space $S(\mathbb{R}^2)$. Thus, the corresponding optical tomogram

$$\omega(X,\phi)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{-itX}F_{\hat{\rho}}(\cos(\phi)t,\sin(\phi)t)dt$$

belongs to $S(\mathbb{R})$ in $X$ and infinitely differentiable in $\phi$. Using the Parseval type equation of Theorem 1

$$\lim_{\varepsilon\to+0}\int\limits_{0}^{2\pi}\int\limits_{-\infty}^{+\infty} w_{\hat{\rho}}(X+i\varepsilon,\phi)a(X+i\varepsilon,\phi)dXd\phi=Tr(\hat{\rho}\hat{a})$$

we can define the extension of dual tomographic map $\hat{a}\to a(X,\phi)$ such that $a(X,\phi)$ should be a generalized function on the set $\Omega$ of optical tomograms $\omega_{\hat{\rho}}$ such that

$$<a,\omega_{\hat{\rho}}>=Tr(\hat{\rho}\hat{a}).$$

$\square$

Proof of Theorem 3.

Given an optical tomogram $\omega_{\hat{\rho}}(X,\phi)$ of a density operator $\hat{\rho}$ we get

$$\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{2\pi} X^{n+m}Q^m_{n+m}(\cos(\phi))\omega_{\hat{\rho}}(X,\phi)dXd\phi=$$

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{2\pi} X^{n+m}Q^m_{n+m}(\cos(\phi))\int\limits_{-\infty}^{+\infty} e^{-iXt}F_{\hat{\rho}}(t\cos(\phi),t\sin(\phi))dtdXd\phi=$$

$$i^n\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{2\pi}\delta^{(n+m)}(t)F_{\hat{\rho}}(t\cos\phi,t\sin(\phi))Q^m_{n+m}(\cos(\phi))dtd\phi=$$

$$(-i)^n\int\limits_{0}^{2\pi}\sum_{k=0}^{n+m}C^k_{n+m}\frac{\partial^{n+m}F_{\hat{\rho}}}{\partial\mu^k\partial\nu^{n+m-k}}(0,0)\cos^k(\phi)\sin^{m+n-k}(\phi)Q^m_{n+m}(\cos(\phi))d\phi=$$

$$(-i)^n\frac{\partial^{n+m}F_{\hat{\rho}}}{\partial\mu^m\partial\nu^n}(0,0).$$

Now the result follows from Proposition 4. $\square$

# Appendix

Let us consider the trigonometric system $\{\sin^k(\phi)\cos^{n-k}(\phi),\ 0\le k\le n\}$. Taking derivatives of $\sin^k(\phi)\cos^{n-k}(\phi)$ give rise to linear combinations of these elements. It follows that $\sin^k(\phi)\cos^{n-k}(\phi)$ satisfy to the linear differential equation of $n+1$th order. Notice that

$$(\sin^k(\phi)\cos^{n-k}(\phi))^{(s)}=0,\ 0\le s<k,\ (\sin^k(\phi)\cos^{n-k}(\phi))^{(k)}=k!,\ if\ \phi=0.$$

Hence the Wronskian $w(0) = \prod_{k=0}^{n} k! \neq 0$ and the elements of this system are linear independent on the segment $[0, 2\pi]$. Thus, there exists the biorthogonal system $\tilde{Q}_n^m(\cos(\phi))$ consisting of polynomials in $\sin^k(\phi)\cos^{n-k}(\phi)$ such that

$$\int_0^{2\pi} \sin^k(\phi)\cos^{n-k}(\phi)\tilde{Q}_n^m(\cos(\phi))d\phi = \delta_{km}.$$

Put $Q_n^m(\cos(\phi)) = \frac{1}{C_n^m}\tilde{Q}_n^m(\cos(\phi))$. The first several polynomials are

$$Q_0^0(\cos(\phi)) = \frac{1}{2\pi},\ Q_1^0(\cos(\phi)) = \frac{1}{\pi}\cos(\phi),\ Q_1^1(\cos(\phi)) = \frac{1}{\pi}\sin(\phi),$$

$$Q_2^0(\cos(\phi)) = -\frac{1}{2\pi}\cos^2(\phi) + \frac{3}{2\pi}\sin^2(\phi),$$

$$Q_2^1(\cos(\phi)) = \frac{2}{\pi}\sin(\phi)\cos(\phi),$$

$$Q_2^2(\cos(\phi)) = \frac{3}{2\pi}\cos^2(\phi) - \frac{1}{2\pi}\sin^2(\phi).$$

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